The Solenoidal Vector Field

1. We of course recall that a **conservative** vector field $\mathbf{C}(\bar{r})$ can be identified from its curl, which is always equal to zero:

$$\nabla \times \mathbf{C}(\bar{r}) = 0$$

Similarly, there is **another** type of vector field $\mathbf{S}(\bar{r})$, called a **solenoidal** field, whose **divergence** is always equal to zero:

$$\nabla \cdot \mathbf{S}(\bar{r}) = 0$$

Moreover, we find that **only** solenoidal vector have zero divergence! Thus, zero divergence is a **test** for determining if a given vector field is solenoidal.

We sometimes refer to a solenoidal field as a **divergenceless** field.

2. Recall that **another** characteristic of a **conservative** vector field is that it can be expressed as the **gradient** of some **scalar** field (i.e., $\mathbf{C}(\bar{r}) = \nabla g(\bar{r})$).

Solenoidal vector fields have a **similar** characteristic! Every solenoidal vector field can be expressed as the **curl** of some other vector field (say $\mathbf{A}(\bar{r})$).

$$\mathbf{S}(\bar{r}) = \nabla x \mathbf{A}(\bar{r})$$

Additionally, we find that **only** solenoidal vector fields can be expressed as the curl of some other vector field. Note this means that:

The curl of **any** vector field **always** results in a solenoidal field!

Note if we **combine** these two previous equations, we get a **vector identity**:

$$\nabla \cdot \nabla x \mathbf{A}(\bar{r}) = 0$$

a result that is always true for **any** and **every** vector field $\mathbf{A}(\bar{r})$.

Note this result is **analogous** to the identify derived from conservative fields:

$$\nabla x \nabla g(\bar{r}) = 0$$

for **all** scalar fields $g(\bar{r})$.

3. Now, let's recall the **divergence theorem**:

$$\iiint_V \nabla \cdot \mathbf{A}(\bar{r})\, dv = \oiint_S \mathbf{A}(\bar{r}) \cdot \overline{ds}$$

If the vector field $\mathbf{A}(\bar{r})$ is **solenoidal**, we can write this theorem as:

$$\iiint_V \nabla \cdot \mathbf{S}(\bar{r})\, dv = \oiint_S \mathbf{S}(\bar{r}) \cdot \overline{ds}$$

But of course, the divergence of a solenoidal field is **zero** ($\nabla \cdot \mathbf{S}(\bar{r}) = 0$)!

As a result, the **left** side of the divergence theorem is zero, and we can conclude that:

$$\oiint_S \mathbf{S}(\bar{r}) \cdot \overline{ds} = 0$$

In other words the **surface** integral of **any** and **every** solenoidal vector field across a **closed** surface is equal to zero.

Note this result is **analogous** to evaluating a line integral of a conservative field over a closed contour

$$\oint_C \mathbf{C}(\bar{r}) \cdot \overline{d\ell} = 0$$

Lets **summarize** what we know about **solenoidal** vector fields:

1. **Every** solenoidal field can be expressed as the **curl** of some **other** vector field.

2. The curl of **any** and **all** vector fields always results in a solenoidal vector field.

3. The **surface integral** of a solenoidal field across any **closed** surface is equal to **zero**.

4. The **divergence** of every solenoidal vector field is equal to **zero**.

5. The divergence of a vector field is zero **only** if it is **solenoidal**.